Transformation of three phase circuits quantities to components

Values in three phase system:

$$\begin{bmatrix} \hat{U}_a \\ \hat{U}_b \\ \hat{U}_c \end{bmatrix} = \begin{bmatrix} \hat{U}_{abc} \end{bmatrix} \qquad \begin{bmatrix} \hat{I}_a \\ \hat{I}_b \\ \hat{I}_c \end{bmatrix} = \begin{bmatrix} \hat{I}_{abc} \end{bmatrix}$$

Values in transformed system

$$\begin{bmatrix} \hat{U}_o \\ \hat{U}_m \\ \hat{U}_n \end{bmatrix} = \begin{bmatrix} \hat{U}_{omn} \end{bmatrix} \qquad \begin{bmatrix} \hat{I}_o \\ \hat{I}_m \\ \hat{I}_n \end{bmatrix} = \begin{bmatrix} \hat{I}_{omn} \end{bmatrix}$$

are defined by a transformation matrixes Tu and Ti:

$$\begin{bmatrix} \hat{U}_{abc} \end{bmatrix} = \begin{bmatrix} \hat{T}_u \end{bmatrix} \begin{bmatrix} \hat{U}_{omn} \end{bmatrix} \qquad \begin{bmatrix} \hat{U}_{omn} \end{bmatrix} = \begin{bmatrix} \hat{T}_u^{-1} \end{bmatrix} \begin{bmatrix} \hat{U}_{abc} \end{bmatrix}$$

$$\begin{bmatrix} \hat{I}_{abc} \end{bmatrix} = \begin{bmatrix} \hat{T}_i \end{bmatrix} \begin{bmatrix} \hat{I}_{omn} \end{bmatrix} \qquad \begin{bmatrix} \hat{I}_{omn} \end{bmatrix} = \begin{bmatrix} \hat{T}_i^{-1} \end{bmatrix} \begin{bmatrix} \hat{I}_{abc} \end{bmatrix}$$

- Tu and Ti must be regular matrixes
- Tu and Ti are usually defined to be equal (Tu=Ti=T)

Derivation of transformation of Z matrix:

Let consider a simple dependence between phase currents and voltages of a three phase circuit element described by a Z matrix.

$$\begin{bmatrix} \hat{U}_a \\ \hat{U}_b \\ \hat{U}_c \end{bmatrix} = \begin{bmatrix} \hat{Z}_{11} & \hat{Z}_{12} & \hat{Z}_{13} \\ \hat{Z}_{21} & \hat{Z}_{22} & \hat{Z}_{23} \\ \hat{Z}_{31} & \hat{Z}_{32} & \hat{Z}_{33} \end{bmatrix} \begin{bmatrix} \hat{I}_a \\ \hat{I}_b \\ \hat{I}_c \end{bmatrix}$$

In matrix formulation:

$$[\hat{U}_{abc}]=[\hat{Z}_{abc}][\hat{I}_{abc}]$$

Similar dependence can be found for transformed voltages and currents

$$[\hat{U}_{omn}]=[\hat{Z}_{omn}][\hat{I}_{omn}]$$

The derivation:

$$[\hat{U}_{abc}]=[\hat{Z}_{abc}][\hat{I}_{abc}]$$

$$[\hat{T}_U][\hat{U}_{omn}]=[\hat{Z}_{abc}][\hat{T}_I][\hat{I}_{omn}]$$

$$[\hat{U}_{omn}]=[\hat{T}_U]^{-1}[\hat{Z}_{abc}][\hat{T}_I][\hat{I}_{omn}]$$

Therefore

$$[\hat{Z}_{omn}] = [\hat{T}_U]^{-1}[\hat{Z}_{abc}][\hat{T}_I]$$

Analogically, it can be found a transformation rule for an admittance matrix:

$$[\hat{Y}_{omn}] = [\hat{T}_I]^{-1}[\hat{Y}_{abc}][\hat{T}_U]$$

- For commonly used transformations: $[\hat{T}_I]=[\hat{T}_U]$
- The formulae for one circuit element can be easily generalized to the whole circuit
- A transformation is a simplification of a problem only if it diagonalises the Z matrix

Which impedance matrixes can be diagonalised?

- Cyclic symmetric matrixes

$$[\hat{Z}_{cs}] = \begin{bmatrix} \hat{Z} & \hat{Z}' & \hat{Z}'' \\ \hat{Z}'' & \hat{Z} & \hat{Z}' \\ \hat{Z}' & \hat{Z}'' & \hat{Z} \end{bmatrix}$$

- Phase symmetric matrixes (a special case of $[\hat{z}_{cs}]$)

$$\left[\hat{Z}_{fs}\right]=\begin{bmatrix}\hat{Z} & \hat{Z}' & \hat{Z}' \\ \hat{Z}' & \hat{Z} & \hat{Z}' \\ \hat{Z}' & \hat{Z}' & \hat{Z}\end{bmatrix}$$

What transformation matrix T diagonalise the Z matrixes?

- A matrix from created from eigen vectors (eigen vectors are columns)

An eigen vector t solves the equation (λ is an eigen value):

$$\left(\left[\hat{Z}_{abc}\right]-\lambda[E]\right)[t]=[0]$$

How to get eigen vectors?

- First get eigen values
- Nonzeros solution for t are possible only if

$$\det\left(\left[\hat{Z}_{abc}\right]-\lambda[E]\right)=0$$

- Solving this equation for $\left[\hat{z}_{fs}\right]$ gives following eigenvalues:

$$\lambda_1 = \hat{Z} + 2\hat{Z}'$$
$$\lambda_2 = \hat{Z} - \hat{Z}'$$
$$\lambda_3 = \hat{Z} - \hat{Z}' = \lambda_2$$

- Using first eigenvalue λ1 in $\left(\hat{Z}_{fs} - \lambda\,[E]\right)[t] = [0]$:

$$\begin{bmatrix} -2\widehat{Z}' & \widehat{Z}' & \widehat{Z}' \\ \widehat{Z}' & -2\widehat{Z}' & \widehat{Z}' \\ \widehat{Z}' & \widehat{Z}' & -2\widehat{Z}' \end{bmatrix} \begin{bmatrix} t_{11} \\ t_{21} \\ t_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gives

$$t_{11} = t_{21} = t_{31}$$

- Using second eigenvalue λ2 in $\left(\hat{Z}_{fs} - \lambda\,[E]\right)[t] = [0]$:

$$\begin{bmatrix} -2\widehat{Z}' & \widehat{Z}' & \widehat{Z}' \\ \widehat{Z}' & -2\widehat{Z}' & \widehat{Z}' \\ \widehat{Z}' & \widehat{Z}' & -2\widehat{Z}' \end{bmatrix} \begin{bmatrix} t_{12} \\ t_{22} \\ t_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gives

$$t_{12} + t_{22} + t_{32} = 0$$

- Using third eigenvalue λ3 in $\left(\hat{Z}_{fs} - \lambda\,[E]\right)[t] = [0]$:

$$
-\begin{bmatrix} -2\widehat{Z}' & \widehat{Z}' & \widehat{Z}' \\ \widehat{Z}' & -2\widehat{Z}' & \widehat{Z}' \\ \widehat{Z}' & \widehat{Z}' & -2\widehat{Z}' \end{bmatrix} \begin{bmatrix} t_{13} \\ t_{23} \\ t_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

- gives

$$
t_{13} + t_{23} + t_{33} = 0
$$

Any transformation matrix [T] which satisfies previous conditions diagonalises $[\hat{z}_{fs}]$.

Transformation to symmetrical components:

Let's remind how a symmetrical three phase voltages can be written

$$
\begin{aligned}
&\hat{U}_a \\
&\hat{U}_b = \hat{a}^2 \hat{U}_a \\
&\hat{U}_c = \hat{a}\,\hat{U}_a \\
&\hat{a} = e^{j\frac{2}{3}\pi} \quad \hat{a}^2 = e^{j\frac{4}{3}\pi} \qquad 1 + \hat{a} + \hat{a}^2 = 0
\end{aligned}
$$

Lets denote for symmetrical components (indexes o,m,n denotes general transformation components):

$$\begin{bmatrix}\hat{U}_o\\ \hat{U}_m\\ \hat{U}_n\end{bmatrix}=\begin{bmatrix}\hat{U}_0\\ \hat{U}_1\\ \hat{U}_2\end{bmatrix}$$

Where

$\hat{U}_0$ is zero component,
$\hat{U}_1$ is positive component,
$\hat{U}_2$ is negative component.

Positive component is defined so that:

$$\begin{bmatrix}\hat{U}_a\\ \hat{a}^2\hat{U}_a\\ \hat{a}\hat{U}_a\end{bmatrix}=\begin{bmatrix}t_{11} & t_{12} & t_{13}\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{33}\end{bmatrix}\begin{bmatrix}0\\ \hat{U}_a\\ 0\end{bmatrix}$$

Which gives

$$t_{12}=1$$
$$t_{22}=\hat{a}^2$$
$$t_{32}=\hat{a}$$

Negative component is defined so that:

$$\begin{bmatrix}\hat{U}_a\\ \hat{a}\hat{U}_a\\ \hat{a}^2\hat{U}_a\end{bmatrix}=\begin{bmatrix}t_{11} & t_{12} & t_{13}\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{33}\end{bmatrix}\begin{bmatrix}0\\ 0\\ \hat{U}_a\end{bmatrix}$$

Which gives

$$t_{13} = 1$$
$$t_{23} = \hat{a}$$
$$t_{33} = \hat{a}^2$$

Zero component is defined so that:

$$\begin{bmatrix} \hat{U}_a \\ \hat{U}_a \\ \hat{U}_a \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} \hat{U}_a \\ 0 \\ 0 \end{bmatrix}$$

Which gives

$$t_{11} = 1$$
$$t_{21} = 1$$
$$t_{31} = 1$$

Therefore

$$\begin{bmatrix} \hat{U}_a \\ \hat{U}_b \\ \hat{U}_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \hat{a}^2 & \hat{a} \\ 1 & \hat{a} & \hat{a}^2 \end{bmatrix} \begin{bmatrix} \hat{U}_0 \\ \hat{U}_1 \\ \hat{U}_2 \end{bmatrix}$$

$$\begin{bmatrix} \hat{T} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \hat{a}^2 & \hat{a} \\ 1 & \hat{a} & \hat{a}^2 \end{bmatrix}$$

$$\begin{bmatrix} \hat{U}_0 \\ \hat{U}_1 \\ \hat{U}_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \hat{a} & \hat{a}^2 \\ 1 & \hat{a}^2 & \hat{a} \end{bmatrix} \begin{bmatrix} \hat{U}_a \\ \hat{U}_b \\ \hat{U}_c \end{bmatrix}$$

$$\left[\hat{T}^{-1}\right] = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \hat{a} & \hat{a}^2 \\ 1 & \hat{a}^2 & \hat{a} \end{bmatrix}$$

Transformation to diagonal components:

Let's denote for symmetrical components:

$$\begin{bmatrix} \hat{U}_o \\ \hat{U}_m \\ \hat{U}_n \end{bmatrix} = \begin{bmatrix} \hat{U}_0 \\ \hat{U}_\alpha \\ \hat{U}_\beta \end{bmatrix}$$

$\hat{U}_0$ is zero component,
$\hat{U}_\alpha$ is component α,
$\hat{U}_\beta$ is component β.

This system of components is suitable as approach to solving of two phase failures.

Zero component is defined so that:

$$\begin{bmatrix} \hat{U}_a \\ \hat{U}_a \\ \hat{U}_a \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} \hat{U}_a \\ 0 \\ 0 \end{bmatrix}$$

Which gives

$$t_{11} = 1$$
$$t_{21} = 1$$
$$t_{31} = 1$$

α, β component is defined so that:

$$\begin{bmatrix} \hat{U}_a \\ \hat{a}\hat{U}_a \\ \hat{a}^2\hat{U}_a \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} 0 \\ \hat{U}_a \\ -j\hat{U}_a \end{bmatrix}$$

Which leads to

$$1 = t_{12} - jt_{13} \Rightarrow t_{12} = 1, \quad t_{13} = 0$$
$$\hat{a}^2 = t_{22} - jt_{23} \Rightarrow t_{22} = -\frac{1}{2}, \quad t_{23} = \frac{\sqrt{3}}{2}$$
$$\hat{a} = t_{32} - jt_{33} \Rightarrow t_{32} = -\frac{1}{2}, \quad t_{33} = -\frac{\sqrt{3}}{2}$$

Therefore

$$\begin{bmatrix} \hat{U}_a \\ \hat{U}_b \\ \hat{U}_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \hat{U}_0 \\ \hat{U}_\alpha \\ \hat{U}_\beta \end{bmatrix}$$

$$[D] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\begin{bmatrix} \hat{U}_0 \\ \hat{U}_\alpha \\ \hat{U}_\beta \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & \sqrt{3} & -\sqrt{3} \end{bmatrix} \begin{bmatrix} \hat{U}_a \\ \hat{U}_b \\ \hat{U}_c \end{bmatrix}$$

$$\left[D^{-1}\right] = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & \sqrt{3} & -\sqrt{3} \end{bmatrix}$$

Transformation form system of diagonal components to system of symmetrical components:

$$\left[\hat{Z}_{012}\right]=\left[T^{-1}\right][D]\left[\hat{Z}_{0\alpha\beta}\right]\left[D^{-1}\right][T]$$

Power invariance:

To ensure power invariance of symmetrical and diagonal components we would have to change the transformation matrixes to (original matrixes devided by $\sqrt{3}$):

$$\left[\hat{T}_n\right]=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \hat{a}^2 & \hat{a}\\ 1 & \hat{a} & \hat{a}^2\end{bmatrix}$$

$$\left[\hat{T}_n^{-1}\right]=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \hat{a} & \hat{a}^2\\ 1 & \hat{a}^2 & \hat{a}\end{bmatrix}$$

$$\left[D_n\right]=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 0\\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2}\\ 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2}\end{bmatrix}$$

$$\left[D_n^{-1}\right] = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & \sqrt{3} & -\sqrt{3} \end{bmatrix}$$

Symmetrical elements of harmonics:

Vector of phasors N (e. g. phase currents or voltages) of k-th harmonics in symmetrical system:

$$\begin{bmatrix} \hat{N}_{aks} \\ \hat{N}_{bks} \\ \hat{N}_{cks} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \hat{a}^2 & 0 \\ 0 & 0 & \hat{a} \end{bmatrix}^k \begin{bmatrix} \hat{N}_{aks} \\ \hat{N}_{aks} \\ \hat{N}_{aks} \end{bmatrix} = \begin{bmatrix} N_{aks} \\ \hat{a}^{2k} N_{aks} \\ a^k N_{aks} \end{bmatrix}$$

Indexes a,b,c – denotes phase
k – harmonics
s – symmetrical system

Dominant symmetrical components for harmonics:

3k	zero component
3k+1	positive component
3k-1	negative component

Let's define for non-symmetrical system:

$$\left[\hat{B}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \hat{b} & 0 \\ 0 & 0 & \hat{c} \end{bmatrix}$$

so that

$$\hat{b} = \frac{\hat{N}_{bk}}{\hat{N}_{bks}}$$

$$\hat{c} = \frac{\hat{N}_{ck}}{\hat{N}_{cks}}$$

Than for symmetrical components of non-symmetrical system and k-th harmonics reads:

$$\begin{bmatrix} \hat{N}_{0k} \\ \hat{N}_{1k} \\ \hat{N}_{2k} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \hat{a} & \hat{a}^2 \\ 1 & \hat{a}^2 & \hat{a} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \hat{b} & 0 \\ 0 & 0 & \hat{c} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \hat{a}^{2k} & 0 \\ 0 & 0 & \hat{a}^k \end{bmatrix}\begin{bmatrix} \hat{N}_{aks} \\ \hat{N}_{aks} \\ \hat{N}_{aks} \end{bmatrix} = \begin{bmatrix} N_{aks} \\ N_{aks} \\ N_{aks} \end{bmatrix}$$