Voltage Distribution along Transformer Winding
Introduction

• Transformers connected to long outdoor lines are often exposed to atmospheric overvoltage
• Should the respective overvoltage protection malfunction, a voltage wave will appear on transformer input terminals
• Such a wave might cause insulation damage and therefore it is necessary to analyze the subsequent voltage transient inside the transformer
Winding behaviour

• Winding is a group of resistors, coils and capacitors
• In the beginning of the transient, the waveform is determined primarily by the capacity of the winding, whereas the inductance governs the waveform at the end
• Let us find the distribution of the voltage during the whole transient
Voltage distribution

- The simplest theory that determines the voltage distribution was presented by Wagner.
- The theory works with the following assumptions:
  - The winding coil has only one layer.
  - The resistance of the winding is minimal, i.e. equal to zero in the calculation.
  - Mutual inductive and capacitive coupling is neglected.
  - A voltage unit step is applied to the input terminals.
Equivalent circuit with distributed parameters

- $L \ (H/m)$ is the total inductance relative to length
- $C \ (F/m)$ is the capacity between the coil wire and the ground relative to length
- $K \ (F\cdot m)$ is inter-turn capacity relative to length
Voltage distribution

• Kirchhoff’s current law for the upper-right node:

\[ i_L + i_K = i_L + \frac{\partial i_L}{\partial x} dx + i_K + \frac{\partial i_K}{\partial x} dx + C dx \frac{\partial u}{\partial t} \quad (1) \]

• Current passing through the longitudinal capacity:

\[ i_K = i_L + \frac{K}{dx} \frac{\partial}{\partial t} \left( u - \frac{\partial u}{\partial x} dx - u \right) = -K \frac{\partial^2 u}{\partial x \partial t} \quad (2) \]

• Kirchhoff’s voltage law for the central loop:

\[ u - \frac{\partial u}{\partial x} dx = L dx \frac{\partial i_L}{\partial t} + u \quad (3) \]
Voltage distribution

• First time derivative of (1) is:

\[ \frac{\partial i_L}{\partial x} + \frac{\partial i_K}{\partial x} + C \frac{\partial u}{\partial t} = 0 \]  (4)

• Applying another time derivative on (1) provides us with:

\[ \frac{\partial^2 i_L}{\partial x \partial t} + \frac{\partial^2 i_K}{\partial x \partial t} + C \frac{\partial^2 u}{\partial t^2} = 0 \]  (5)

• The first term can be substituted from (2) as:

\[ \frac{\partial^2 i_L}{\partial x \partial t} - K \frac{\partial^4 u}{\partial x^2 \partial t^2} + C \frac{\partial^2 u}{\partial t^2} = 0 \]  (6)
Voltage distribution

• By substituting from (3), we receive next:

\[- \frac{1}{L} \frac{\partial^2 u}{\partial x^2} - K \frac{\partial^4 u}{\partial x^2 \partial t^2} + C \frac{\partial^2 u}{\partial t^2} = 0\]  \hspace{1cm} (7)

• Multiplying the previous equation by \(-L\) gives us the ultimate expression:

\[\frac{\partial^2 u}{\partial x^2} + LK \frac{\partial^4 u}{\partial x^2 \partial t^2} - LC \frac{\partial^2 u}{\partial t^2} = 0\]  \hspace{1cm} (8)

• In other words, we obtained a wave equation for a coil with unknown function of voltage \(u(t,x)\)

• Since (8) is a fourth order partial differential equation, its analytical solution will be rather difficult

• Such equations are generally solved by numerical methods
Voltage distribution – solution

• Let us focus on the analytical solution of the two simplest cases, i.e., for $t = 0$ and $t \to \infty$

• In the case $t = 0$, inductance effectively behaves as an open circuit, allowing no current to pass through it

• Therefore, we can omit inductances in the schematic and work only with capacitances

• Also, since time is constant, the function of voltage reduces to $u(0,x) = u_0$
Voltage distribution – solution for $u_0$

- The first step of deriving the expression for $u_0$ utilizes Kirchhoff’s current law. Since time is fixed, the current transforms into charge, as follows:

$$q_0 - (q_0 + dq_0) - Cdxu_0 = 0,$$  \hspace{1cm} (9)

where $q_0$ is the charge on capacity $K/dx$

- In (9), $q_0$ vanishes after subtraction. If we multiply the equation by $1/dx$, we obtain:

$$-\frac{dq_0}{dx} = Cu_0$$  \hspace{1cm} (10)
Voltage distribution – solution for $u_0$

- Kirchhoff’s law for voltage loop provides us with:
  \[ u_0 - du_0 = \frac{q_0}{K} + u_0 \]  
  \[ (11) \]

- Charge $q_0$ can be expressed from (11) as:
  \[ q_0 = -K \frac{du_0}{dx} \]  
  \[ (12) \]

- By deriving (12) by time, we receive the following expression:
  \[ \frac{dq_c}{dx} = -K \frac{d^2u_0}{dx^2} = -C u_0 \]  
  \[ (13) \]

- Finally, by substituting the right-hand term with (10), we obtain:
  \[ \frac{d^2u_0}{dx^2} = \frac{C}{K} u_0 \]  
  \[ (14) \]
Voltage distribution – solution for $u_0$

• Equation (14) represents a second order ordinary differential equation with solution in form:

$$u_0 = A_0 e^{\gamma x} + B_0 e^{-\gamma x} \quad (15)$$

where $\gamma = \sqrt{C/K}$

Coefficients $A_0$ and $B_0$ can be determined from boundary conditions, i.e., $u_0(x = 0) = 1$ and $u_0(x = l) = 0$ (for grounded end of winding of total length $l$):

$$A_0 + B_0 = 1 \rightarrow B_0 = 1 - A_0 \quad (16)$$

$$A_0 e^{\gamma l} + B_0 e^{-\gamma l} = 0 \quad (17)$$
Voltage distribution – solution for $u_0$

- Substituting (16) into (17) gives us:
  $$A_0 e^{\gamma l} + (1 - A_0) e^{-\gamma l} = 0 \rightarrow$$
  $$A_0 = \frac{-e^{-\gamma l}}{e^{\gamma l} - e^{-\gamma l}}, \quad B_0 = \frac{e^{\gamma l}}{e^{\gamma l} - e^{-\gamma l}} \quad (18)$$

- Using the terms from (18), we obtain new equation for $u_0$ as per (15):
  $$u_0 = \frac{-e^{-\gamma l}}{e^{\gamma l} - e^{-\gamma l}} e^{\gamma x} + \frac{e^{\gamma l}}{e^{\gamma l} - e^{-\gamma l}} e^{-\gamma x} \quad (19)$$
Voltage distribution – solution for $u_0$

- By converting right-hand side of (19) into one term, we receive the following expression:

$$u_0 = \frac{e^\gamma(l-x) - e^{-\gamma(l-x)}}{e^{\gamma l} - e^{-\gamma l}} \quad (20)$$

- Equation (20) can be rewritten using hyperbolic function identity:

$$\sinh x = \frac{1}{2} (e^x - e^{-x}), \quad (21)$$

as

$$u_0 = \frac{\sinh \gamma(l-x)}{\sinh \gamma l} \quad (22)$$
Voltage distribution – solution for $u_F$

- Let us examine the other case, where time $t$ approaches infinity (we search for final values of voltage $u_F$)
- As the transient certainly becomes steady state for times approaching infinity, we assume that all functions are independent on time, and hence their time derivatives are zero
- Applying the previous to equation (8), we obtain:

$$\frac{d^2 u_F}{dx^2} = 0 \quad (23)$$
Voltage distribution – solution for $u_F$

• The solution of (23), which is an implicit ordinary differential equation of second order, is in form:

$$u_F = A_F \cdot x + B_F \quad (24)$$

• Once again, $A_F$ and $B_F$ is determined by boundary conditions (grounded winding):

$$u_F(0) = 1 \rightarrow B_F = 1 \quad (25)$$

and

$$u_F(l) = 0 \rightarrow A_F \cdot l + 1 = 0 \rightarrow A_F = -\frac{1}{l} \quad (26)$$
Voltage distribution – maximum stress

- Undamped EM oscillations occur between $u_0$ and $u_F$. Their trend is given by the solution of equation (8) (original PDE) for respective times $t$
- Winding insulation is stressed the most at time $t = 0$ and position $x = 0$. This can be expressed as:

$$E_0 = -\text{grad}(u_0) = -\frac{du_0}{dx} = -\frac{d}{dx}\left(\frac{\sinh(\gamma(l-x))}{\sinh(\gamma l)}\right) = \frac{\gamma \cosh(\gamma(l-x))}{\sinh(\gamma l)} \to E_{0,\text{max}} = \gamma \cdot \cotgh(\gamma l)$$ (27)
- Practically $\gamma l > 3$, therefore $\cotgh(\gamma l) \approx 1$ and $E_{0,\text{max}} \approx \gamma$
Voltage distribution – ungrounded winding

• Initial voltage distribution along an ungrounded coil for different values of $\gamma l$

• The derivation of expression for $u_0$ for ungrounded coil is more complicated, as the second boundary condition is unknown ($u_0(0,l) \neq 0$)

• Therefore, let us state only the ultimate expression:

$$u_0 = \frac{\cosh \gamma (l-x)}{\cosh \gamma l}$$  \hspace{1cm} (28)
Voltage distribution plots

• Initial voltage distribution along a grounded and ungrounded coil for different values of $\gamma$
Voltage distribution – effects

• During the transient, undamped oscillations can reach values of up to 150/280 % of the initial voltage for grounded/ungrounded winding, respectively.

• To prevent such large values, several methods of equalizing the initial voltage distribution along the winding are employed:
  – Disc windings: ground capacity $C$ can be compensated (capacitive screen) and/or the series capacity can be increased (turn interlacing)
  – Multi-layered windings: ground capacity $C$ is present only for the first and the last layer. Moreover, the inter-layer capacity is much larger than the capacity between winding discs.