Freely-Hanging electric power cable

Used variables:

d – diameter of wire (mm),

S – cross-section of wire (mm^2),

q_1 – weight of wire per 1 m length ($N.m^{-1}$),

γ - specific weight ($MPa.m^{-1}$),

q_2 – weight of additional load per 1 m ($N.m^{-1}$),

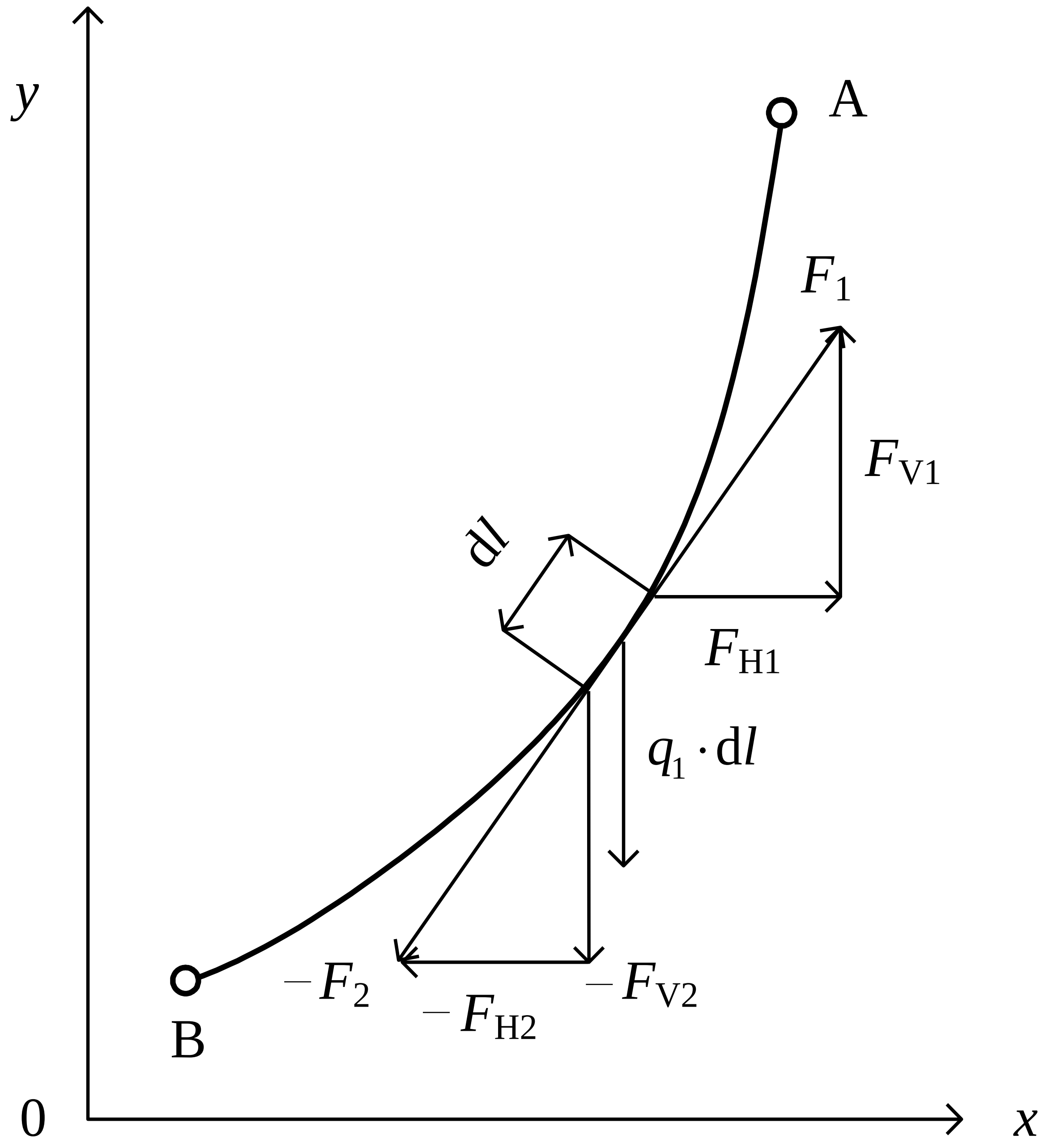
z – wire overload,

σ - wire stress (MPa),

σ_H – horizontal component of stress (MPa),
σ_V - vertical component of stress (MPa),
F – force (tensile) in wire (N),
F_H - horizontal component of force in wire (N),
F_V - vertical component of force in wire (N),
l – wire length (m),
α - span (m),
f – sag (m),
c - parameter (m),
ϑ - temperature (°C),
α - coefficient of thermal expansion (K^{-1}),
E – Young modulus ($N.mm^{-2}$).

1. Horizontal span

-suspension points at the same heights



Obr. 1.1.

Inelastic catenary x elastic catenary?

- Only inelastic catenary will be considered

Derivation of catenary shape

For derivation of formula, that describes its shape, we will use two facts:

1. Sum of forces on an element of wire must be zero

dl – length of the element $\mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$

$d\bar{F}$ – difference of forces on both sides of the element $d\bar{F} = \bar{F}_1 - \bar{F}_2$

dF_H – difference of vertical forces on both sides of the element $dF_H = F_{H1} - F_{H2}$

dF_V – difference of vertical forces on both sides of the element $dF_V = F_{V1} - F_{V2}$

Vector equation

$$d\bar{F} = \bar{q}_1 \cdot dl$$

Rewritten to horizontal and vertical compound:

$$dF_H = 0$$

$$dF_V = q_1 \cdot dl$$

First equation says that horizontal compound of force is constant along the wire.
The equation for vertical compound of force can be divided by dx and rewritten as (it will be used little later)

$$\frac{\mathrm{d}F_\mathrm{V}}{\mathrm{d}x} = q_1 \cdot \frac{\mathrm{d}l}{\mathrm{d}x} = q_1 \sqrt{\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\mathrm{d}x^2}} = q_1 \sqrt{1 + y'^2}$$

2. Force on an element of wire is of tangential direction to the wire:

$$F_\mathrm{V} = F_\mathrm{H} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$$

- By combination of last two equations:

$$\frac{\mathrm{d}F_\mathrm{V}}{\mathrm{d}x} = F_\mathrm{H} \cdot \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = q_1 \sqrt{1 + y'^2}$$

After rearrangement we will get the final equation:

$$\sqrt{1+y'^2} = \frac{F_\text{H}}{q_1} \cdot \frac{\text{d}^2 y}{\text{d}x^2}$$

where

$$\frac{F_\text{H}}{q_1} = \frac{F_\text{H}}{S} \cdot \frac{S}{q_1} = \frac{\sigma_\text{H}}{\gamma}$$

is a constant.

To solve the equation, we will introduce following in it:

$$\frac{F_\text{H}}{q_1} = \frac{\sigma_\text{H}}{\gamma}$$

$$\frac{\text{d}^2 y}{\text{d}x^2} = \frac{\text{d}y'}{\text{d}x}$$

and get:

$$\frac{\sigma_\text{H}}{\gamma} \cdot \frac{\text{d}y'}{\text{d}x} = \sqrt{1+y'^2}$$

This is a differential equation between y` and x. We will rearrange it so that y` is on one side, x on another and integrate it:

$$\frac{\mathrm{d}y'}{\sqrt{1+y'^2}} = \frac{\gamma}{\sigma_\mathrm{H}}\mathrm{d}x$$

$$\int \frac{\mathrm{d}y'}{\sqrt{1+y'^2}} = \int \frac{\gamma}{\sigma_\mathrm{H}}\mathrm{d}x$$

The solution of the integration is:

$$\operatorname{arg\,sinh}\ y' = \frac{\gamma}{\sigma_\mathrm{H}}x + k$$

Therefore, we can write

$$y' = \sinh\frac{\gamma}{\sigma_\mathrm{H}}\left(x + k_1\right)$$

By integration of this equation we obtain the general formula of catenary:

$$y = \frac{\sigma_\mathrm{H}}{\gamma}\cosh\frac{\gamma}{\sigma_\mathrm{H}}\left(x + k_1\right) + k_2$$

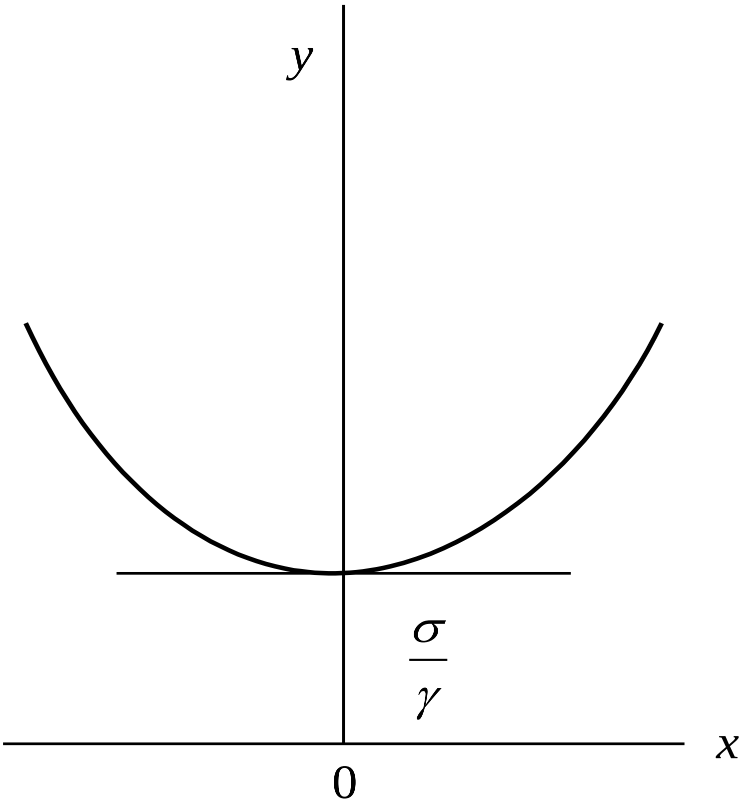
The constants k_1 and k_2 can be found using co-ordinates of suspension points.

However, we will use an opposite approach: we will find and us such a coordinate system, that k_1 and k_2 would be zero.

First condition for such a coordinate system is:

- $x = 0$ for $y' = 0$
  because than

$$0 = \sinh \frac{\gamma}{\sigma_{\mathrm{H}}} k_1 \qquad k_1 = 0$$



Obr. 1.2.

- Second condition is $y = \dfrac{\sigma_H}{\gamma}$ for $x = 0$

  because than:

$$\frac{\sigma_H}{\gamma} = \frac{\sigma_H}{\gamma} + k_2 \qquad k_2 = 0$$

Equation of cunicular for such a coordinate system than reads:

$$y = \frac{\sigma_{\mathrm{H}}}{\gamma} \cdot \cosh \frac{x}{\frac{\sigma_{\mathrm{H}}}{\gamma}} \quad (\mathrm{m})$$

Let's yet denote

$$\frac{\sigma_{\mathrm{H}}}{\gamma} = c$$

And we will get the equation of catenary in final form

$$y = c \cdot \cosh \frac{x}{c} \quad (\mathrm{m})$$

Catenary equation application notes

- It is possible to take into account a continuous additional load q_2 by changing the definition of the constant c

$$c = \frac{F_{\mathrm{H}}}{q_1 + q_2} = \frac{F_{\mathrm{H}} \cdot q_1}{q_1 \left(q_1 + q_2\right)} = \frac{F_{\mathrm{H}}}{q_1\, z} \cdot \frac{S}{S} = \frac{\sigma_{\mathrm{H}}}{\gamma z} \quad (\mathrm{m})$$

- When applying Taylor series to the equation we get:

$$y = c\left[1 + \frac{1}{2!}\left(\frac{x}{c}\right)^2 + \frac{1}{4!}\left(\frac{x}{c}\right)^4 + \frac{1}{6!}\left(\frac{x}{c}\right)^6 + ...\right]$$

Taking into account only the first two members we will get a parabola.

$$y = c + \frac{x^2}{2c} = \frac{\sigma_H}{\gamma z} + \frac{x^2}{2}\frac{\gamma z}{\sigma_H}$$

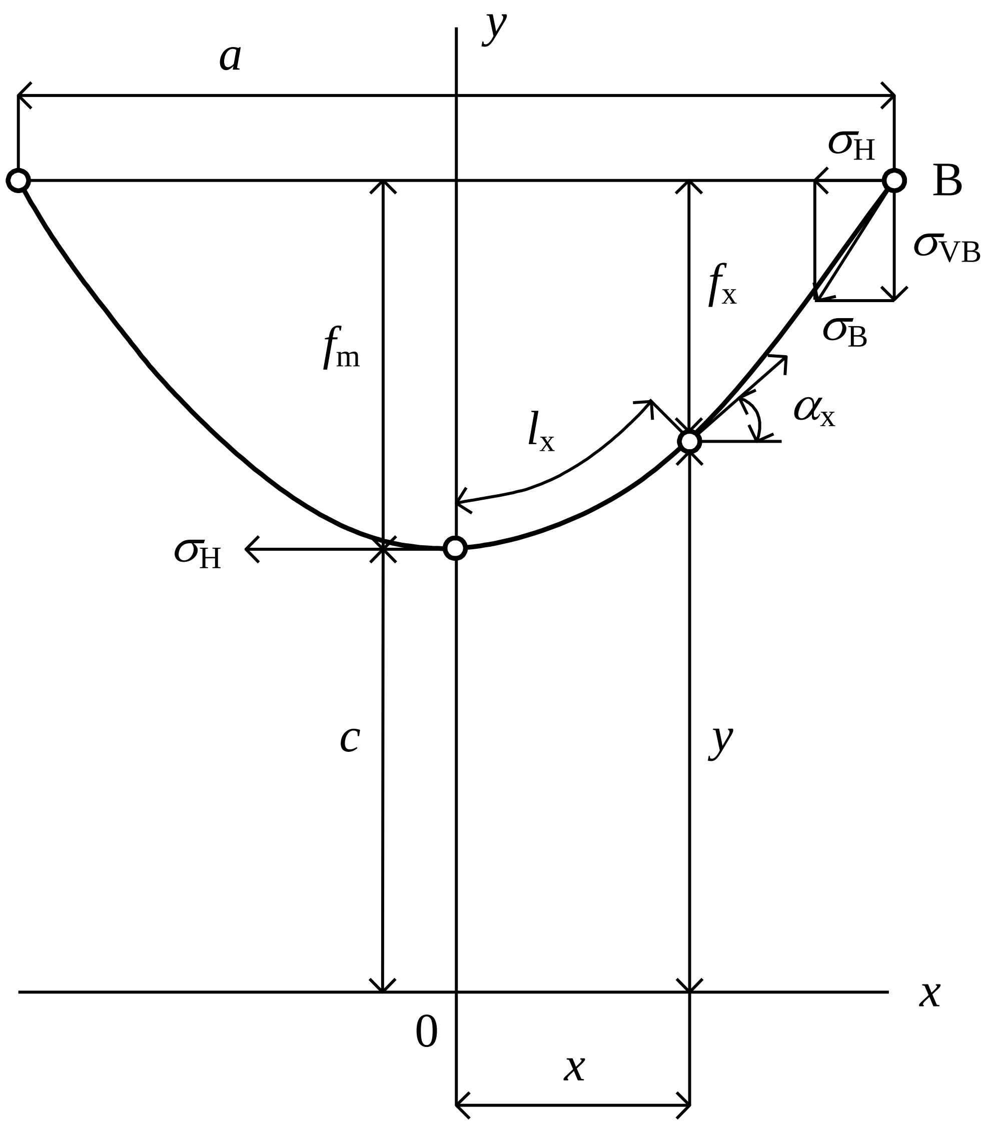
It is a good approximation, very often used.

- How to get maximal sag f_m?
  Set half of the span length for x:

$$f_m = y(a/2) - y(0)$$

and get:

$$f_m = c\left(\cosh\frac{a}{2c} - 1\right)$$



Obr. 1.3. Symmetrical catenary

- How to get sag f_x in an arbitrary point x?

$$f_x = y(a/2) - y(x)$$

$$f_x = c\left(\cosh\frac{a}{2c} - \cosh\frac{x}{c}\right)$$

- How to get length of the wire?

$$l_s = 2\int_0^{\frac{a}{2}} \sqrt{1 + y'^2}\,\mathrm{d}x$$

$$1 + y'^2 = \cosh^2 \frac{x}{c}$$

$$l_s = 2\int_0^{\frac{a}{2}} \cosh \frac{x}{c}\,\mathrm{d}x = 2c\left[\sinh \frac{x}{c}\right]_0^{\frac{a}{2}} = 2c \cdot \sinh \frac{a}{2c}$$